If a is half an odd integer, as in the case of the Mathieu functions ($a = -{}^1/_2$), tan $a\pi$ in (27) becomes infinite. The function $(cz)^{-a-1/2}\sum_n{}' a_n^l J_{-n-a-1/2}(cz)$, however, is then no longer independent of $Re^1_{a,l}$ and, as might be expected, it is easy to show that $Se^2_{a,l}$ is now proportional to $Se^1_{a,l}$. An independent expression for $Se^2_{a,l}$ applicable in this case may be obtained by a limiting process, the details of which will be published later.

[1] J. A. Stratton, "Spheroidal Functions," *Proc. Nat. Acad. Sci.*, **21**, 51–56 (1935).
[2] E. W. Hobson, *Theory of Spherical and Ellipsoidal Harmonics*, p. 193.
[3] G. N. Watson, *Theory of Bessel Functions*, p. 164.
[4] Whittaker and Watson, *Modern Analysis*, 4th Ed., p. 245.
[5] Hobson, loc. cit., p. 229.

THERMAL EQUILIBRIUM IN A GENERAL GRAVITATIONAL FIELD

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1. *Introduction.*—In the absence of any appreciable gravitational field the conditions satisfied by the temperature of a medium having no thermal flow from one portion to another, are well known to be those of uniformity throughout the system. In the presence of a gravitational field, however, the conditions imposed at thermal equilibrium on the proper temperature—as measured at different points of the medium by local observers—are considerably more complicated, owing to the association of inertia and weight with all forms of energy and the consequent tendency for heat to flow from regions of higher to those of lower gravitational potential. Nevertheless, for the special case of static gravitational fields, these more complicated conditions have already been obtained,[1,2] and it is the purpose of the present article to investigate the conditions for thermal equilibrium in the general case of any kind of gravitational field.

The investigation shows that a reasonably straightforward treatment can be found which leads to an apparently satisfactory, covariant expression of the general conditions for thermal equilibrium. The possible value of such a result can be two-fold. In the first place it satisfies our desire for coherent theory, since our previous expression of the conditions for thermal equilibrium, which applied only to static systems and was made in the non-covariant language of a kind of coördinate system then appropriate, can now be regarded as a special case of the generally covariant expression applying to any kind of system. In the second place, the expression obtained might prove useful if it should sometime become necessary to consider non-homogeneous cosmological models with thermal flow from one portion of the model to another.

2. *Conditions for Thermal Equilibrium in a Static System.*—Let us first consider the case of a static system corresponding to the line element

$$ds^2 = g_{ij}dx^i dx^j + g_{44}dx^4 dx^4 \qquad (i, j = 1, 2, 3) \tag{1}$$

where all the components of the metrical tensor are independent of the time-like coördinate x^4

$$\frac{\partial g_{\mu\nu}}{\partial x^4} = 0 \qquad (\mu, \nu = 1, 2, 3, 4) \tag{2}$$

and the components of the "velocity" of the medium are given by

$$u^i = \frac{dx^i}{ds} = 0, \qquad u^4 = \frac{dx^4}{ds} = (g^{44})^{1/2}. \tag{3}$$

The conditions at thermal equilibrium are then known to be given by the expressions

$$\frac{\partial \log T_0}{\partial x^i} = -\frac{\partial \log (g_{44})^{1/2}}{\partial x^i} \tag{4}$$

$$\frac{\partial \log T_0}{\partial x^4} = 0 \tag{5}$$

where the first equation relates the proper temperature T_0—as measured at different positions by local observers at rest in the medium—to the gravitational potential g_{44}, and the second equation states that the proper temperature at a given position is independent of the time-like coördinate x^4 in accordance with our assumption of a static system.

The above expression for the relation between proper temperature and position in the gravitational field has received a satisfactory derivation by considering the effect of the field on the pressure and hence temperature of black-body radiation in equilibrium with the system,[1] and—for the analytically simple case of a fluid system with spherical symmetry—it has also been justified by direct use of the relativistic form of the second law of thermodynamics.[2] Hence equations (4) and (5) should provide a reliable starting point for further generalization.

3. *Conditions for Thermal Equilibrium in a Static System Reëxpressed in Covariant Language.*—As a start in the process of generalization, we may first reëxpress the above conditions for thermal equilibrium in the case of static systems in the simple form

$$\frac{\partial \log T_0}{\partial x^\nu} = (u_\nu)_\alpha u^\alpha \quad (\nu, \alpha = 1, 2, 3, 4) \tag{6}$$

where u_ν and u^α refer to the "velocity" of the medium with respect to the coördinates used. This expression can also be written in the more explicit forms

$$\frac{\partial \log T_0}{\partial x^\nu} = g_{\nu\beta}(u^\beta)_\alpha u^\alpha$$

$$= g_{\nu\beta}\left(\frac{\partial u^\beta}{\partial x^\alpha} + \{\alpha\gamma, \beta\} u^\gamma\right)u^\alpha$$

$$= g_{\nu\beta}\left(\frac{\partial u^\beta}{\partial x^\alpha} + \frac{1}{2} g^{\beta\delta}\left\{\frac{\partial g_{\ \alpha}}{\partial x^\gamma} + \frac{\partial g_{\gamma\delta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\gamma}}{\partial x^\delta}\right\}u^\gamma\right)u^\alpha. \qquad (6')$$

Since equation (6) is seen to be a tensor equation, it will be true in all coördinate systems if true in one, and will hence be a valid covariant expression of the conditions for thermal equilibrium in static systems, if it reduces to the former equations (4) and (5) when expressed in the special coördinates that were then used. In these coördinates, however, we know from (3) that u^γ and u^α will be zero except for γ and α equal to 4, and from the static character of the system in these coördinates we know that all derivatives with respect to x^4 will vanish. Hence our equation in the last form (6′), immediately reduces to

$$\frac{\partial \log T_0}{\partial x^\nu} = -\frac{1}{2} g_{\nu\beta} g^{\beta\delta} \frac{\partial g_{44}}{\partial x^\delta} u^4 u^4$$

$$= -\frac{1}{2} g_\nu^\delta \frac{\partial g_{44}}{\partial x^\delta} g^{44} = -\frac{\partial \log (g_{44})^{1/2}}{\partial x^\nu}$$

and with $\nu = 1, 2, 3$ this is equivalent to (4), while that $\nu = 4$ it is equivalent to (5) on account of the static character of the system. Thus equation (6) is shown to be a valid covariant expression of the conditions for thermal equilibrium in static systems.

4. *Conditions for Thermal Equilibrium in a Static System Reëxpressed in Proper Coördinates.*—In accordance with the foregoing, nevertheless, equation (6) is presumably an expression of the conditions for thermal equilibrium only in the case of static systems, and some further physical content may have to be included to obtain an expression applicable to the general case of non-static systems, which will be of such nature that no coördinates can be found that will permit a description of the system by our original equations (1), (2) and (3).

To obtain an idea of the physical content already implied by equation (6), we shall find it profitable to reëxpress its consequences in terms of the proper coördinates (x, y, z, t) which would be appropriate for use by a local observer momentarily at rest with respect to the medium at some selected point and time. In these proper coördinates, the expression for the line element will reduce in the immediate neighborhood of that point and time to the special relativity form

$$ds^2 = -dx^2 - dy^2 - dz^2 + dt^2 \tag{7}$$

where the gravitational potentials have their Galilean values and their first derivatives vanish

$$g_{\mu\nu} = \pm 1, 0 \qquad \frac{\partial g_{\mu\nu}}{\partial x^\alpha} = 0. \tag{8}$$

Furthermore, in proper coördinates the components of the "velocity" of the medium will have the simple values

$$u^i = \frac{dx^i}{ds} = 0 \qquad (i = 1, 2, 3) \qquad u^4 = \frac{dt}{ds} = 1. \tag{9}$$

Finally, since we can write the general formula for interval in the form

$$1 = g_{11}\frac{dx}{ds}\frac{dx}{ds} + 2g_{12}\frac{dx}{ds}\frac{dy}{ds} + \ldots\ldots + g_{44}\frac{dt}{ds}\frac{dt}{ds}$$

it is evident that we must have as a consequence of differentiating both sides of this expression the relation

$$\frac{\partial}{\partial x^\alpha}\left(\frac{dt}{ds}\right) = \frac{\partial u^4}{\partial x^\alpha} = 0 \tag{10}$$

at the point of interest, since the differentiation of all terms in the above expression except the last will lead to zero after (8) and (9) are substituted.

Making use of equations (8), (9) and (10) in connection with the conditions for equilibrium given by (6′), we then easily obtain

$$\left.\begin{aligned}
\frac{\partial \log T_0}{\partial x} &= -\frac{\partial u^1}{\partial t} = -\frac{\partial u_x}{\partial t} = -\dot{u}_x \\
\frac{\partial \log T_0}{\partial y} &= -\frac{\partial u^2}{\partial t} = -\frac{\partial u_y}{\partial t} = -\dot{u}_y \\
\frac{\partial \log T_0}{\partial z} &= -\frac{\partial u^3}{\partial t} = -\frac{\partial u_z}{\partial t} = -\dot{u}_z
\end{aligned}\right\} \tag{11}$$

$$\frac{\partial \log T_0}{\partial t} = 0 \tag{12}$$

as an expression of the conditions for thermal equilibrium in a static system using proper coördinates for the selected point and time, where it will be noted that $\dot{u}_x$, $\dot{u}_y$, $\dot{u}_z$ are the components of the acceleration of the medium as it would be observed and measured in the ordinary manner by the local observer at the point of interest.

This result makes the physical implications of our present conditions for thermal equilibrium fairly clear. Equations (11) show that the local freely falling observer—for whom gravitational effects have been abolished—will find a temperature gradient oppositely directed to the acceleration which he observes for the medium, and in accordance with the principle that all forms of energy have inertia will interpret this as the necessary condition to prevent heat from "lagging behind" in the accelerated medium. Equation (12) on the other hand merely states that the temperature at a given point in the medium is not observed to be changing with the time in agreement with our assumption of a static system.

5. *Conditions for Thermal Equilibrium in a General System.*—The above result also makes clear what further physical content should be included in order to treat non-static as well as static systems. Again considering proper coördinates for a particular point in the medium, it seems evident from the above mentioned notions as to the relation between energy and inertia that equations (11) should still be retained as the conditions to prevent thermal flow through the medium. On the other hand, it also appears evident that equation (12) is too restrictive for the general case, since in a non-static system even in the absence of heat flow the temperature at a given point in the medium might be changing with proper time, owing to the local "generation of heat" by a variety of processes of which adiabatic compression would be a simple example.

These considerations then immediately suggest, as a tensor expression of the general conditions for the absence of thermal flow in any kind of system, the equation

$$\begin{aligned}\frac{\partial \log T_0}{\partial x^\nu} &= (u_\nu)_\alpha u^\alpha + u_\nu \frac{d \log T_0}{ds} \\ &= g_{\nu\beta}\left(\frac{\partial u^\beta}{\partial x^\alpha} + \{\alpha\gamma, \beta\} u^\gamma\right) u^\alpha + g_{\nu\beta} u^\beta \frac{d \log T_0}{ds}. \qquad (13)\end{aligned}$$

With the help of equations (8), (9) and (10), this expression is easily seen to reduce in proper coördinates at the point of interest for the cases $\nu = 1, 2, 3$ to equations (11), and for the case $\nu = 4$ to the identity

$$\frac{\partial \log T_0}{\partial t} = \frac{d \log T_0}{ds} \qquad (14)$$

which allows for any arbitrary changes in the local temperature of the medium which actually do take place for causes other than the flow of heat. Since equation (13) is a covariant expression its validity in proper coordinates secures its validity in all coördinates, and we may hence regard it as a satisfactory expression of the conditions for thermal equilibrium in a general gravitational field.

[1] Tolman and Ehrenfest, *Phys. Rev.*, **36**, 1791 (1930).
[2] Tolman, *Ibid.*, **35**, 904 (1930).

ON THE EXPANSION OF GREEN'S FUNCTION

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In the case of a number of inhomogeneous partial differential equations the appropriate Green's function is known, and this knowledge gives a theoretical means of solving the equation; though in practice not much can be done unless an expansion of the Green's function is known. It is the purpose of the present paper to indicate a general method for finding such expansions.

This is possible only if the differential equation is separable; for this reason some specification as to its form is essential. Confining our interest to scalar equations of mathematical physics, we note that all of these which consider isotropic media in Euclidian space, and are of order less than four in the space derivatives, necessarily involve Laplace's operator and can be written in the form

$$\nabla^2\varphi + f\left(x_1x_2x_3\frac{\partial}{\partial t}\right)\varphi = \rho(x_1x_2x_3t). \tag{1}$$

When the h_i that determine the line element

$$ds^2 = h_1^2dx_1^2 + h_2^2dx_2^2 + h_3^2dx_3^2 \tag{2}$$

have certain functional forms and f satisfies certain conditions, the homogeneous equation corresponding to (1) can be separated.[1] We show here that the problem of fitting together the resulting solutions so as to form a solution of (1) which satisfies the same boundary conditions at infinity as the Green's function can be made to depend on the solution by Green's method of a single inhomogeneous total differential equation. This last is quite easy and it remains only to compare the solution of (1) thus found with that obtained from the Green's function and so to find the desired expansion.

In showing how the above described process can be carried out we will give the analysis for Poisson's equation in three variables; the principle